Discrepancies Between Euclidean and Spherical Trigonometry

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Non-Euclidean geometry is geometry that is not based on the postulates of Euclidean geometry. The five postulates of Euclidean geometry are:

1. Two points determine one line segment.
2. A line segment can be extended infinitely.
3. A center and radius determine a circle.
4. All right angles are congruent.
5. Given a line and a point not on the line, there exists exactly one line containing the given point parallel to the given line.

The fifth postulate is sometimes called the parallel postulate. It determines the curvature of the geometry's space. If there is one line parallel to the given line (like in Euclidean geometry), it has no curvature. If there are at least two lines parallel to the given line, it has a negative curvature. If there are no lines parallel to the given line, it has a positive curvature. The most important non-Euclidean geometries are hyperbolic geometry and spherical geometry.

Hyperbolic geometry is the geometry on a hyperbolic surface. A hyperbolic surface has a negative curvature. Thus, the fifth postulate of hyperbolic geometry is that there are at least two lines parallel to the given line through the given point.

Spherical geometry is the geometry on the surface of a sphere. The five postulates of spherical geometry are:

1. Two points determine one line segment, unless the points are antipodal (the endpoints of a diameter of the sphere), in which case they determine an infinite number of line segments.
2. A line segment can be extended until its length equals the circumference of the sphere.
3. A center and a radius with length less than or equal to πr where r is the radius of the sphere determine a circle.
4. All right angles are congruent.
5. Given a line and a point not on the line, there are no lines through the given point parallel to the given line.

A line segment on a sphere is the shortest distance between two points on the sphere. This distance is the smaller section of arc formed by the two points on a great circle. A great circle is a circle whose center is the center of the sphere and whose radius is the radius of the sphere.

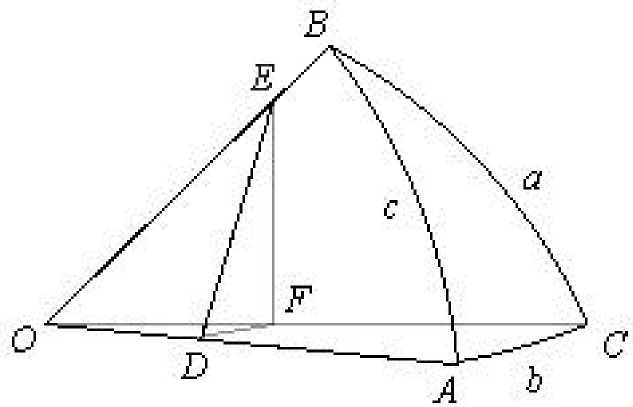
In spherical geometry, a triangle is the section of a sphere bounded by the arcs of three great circles. There are two sections formed in this way. One is the section whose interior angles each have a measure less than 180°. The other is the section whose interior angles each have a measure greater than 180°. I will be working only with the first triangle, and will regard it as *the* spherical triangle formed by the arcs of three great circles. Also, each side must have a length less than or equal to half the circumference of the sphere because otherwise the side would not fit the definition of a line segment.

I will focus on the following problem: For a triangle ABC — where *a*, *b*, and *c* are the lengths of the sides opposite angles A, B, and C, respectively —, given *a, b,* and the measure of angle C, what is the relationship between *c* where ΔABC is in a Euclidean plane and *c* where ΔABC is on the surface of a sphere, and what is the relationship between the area of ΔABC in a Euclidean plane and ΔABC on the surface of a sphere? In other words, find $\frac{c_{\text{Euclidean}}}{c_{\text{spherical}}}$ and $\frac{\text{Area}_{\text{Euclidean}}}{\text{Area}_{\text{spherical}}}$. The values of these ratios will vary depending on the size and shape of the triangles, as well as how large they are relative to the sphere. I will explore these ratios for a variety of circumstances.

To do this, I will use spherical trigonometry. Because ΔABC is not necessarily right, I will use trigonometry for oblique triangles. The key formulae for solving this problem are the law of cosines for sides and the law of sines. The derivations of them are much like the derivations for the law of cosines and law of sines in Euclidean trigonometry. The first step is to derive a trigonometry for right triangles.

The measure of the angle formed by the intersection of two great circles is defined to be equal to the measure of the angle formed by the two lines tangent to each great circle at the point of intersection. It is also equal to the measure of the dihedral angle formed by the planes of the great circles. (The measure of a dihedral angle is the measure of the angle formed by the two lines of intersection made by the two planes forming the dihedral angle with a plane perpendicular to the two planes' line of intersection.) This is shown in the following proof.

Call the point of intersection of the two great circles A and the center of the sphere O. Let lines *l* and *m* each be tangent at A to one of the great circles that intersect at A. *l* OA and *m* OA because *l* and *m* are each tangent to a circle that OA is a radius of (and a line tangent to a circle is perpendicular to the radius of the circle that intersects it). Because of this, the measure of the dihedral angle formed by planes *l*-O and *m*-O is equal to the measure of the angle formed by *l* and *m*, and therefore equal to the measure of the spherical angle formed by the intersection of the great circles.



[1]

In the picture above, O is the center of the sphere. ΔABC is a right spherical triangle with the right angle at C.

Construct plane DEF through any point E on OB such that plane DEF OA. Let D be on OA, and F be on OC. DE OA and DF OA because DEF OA and DE and DF are on DEF. Then ΔODF is a right triangle with its right angle at D because DF OA, and ΔODE is a right triangle with its right angle at D because DE OA. DEF OAC because DEF OA and OA is on OAC. If two planes intersect and are each perpendicular to a third plane then their line of intersection is perpendicular to the third plane; so EF OAC because

BCO OCA and DEF OCA. OC is on OCA, therefore EF OC, so ΔOFE is right. Also, DF is on OCA, therefore EF DF, so ΔDFE is right.

For each part of spherical triangle ABC, there is an angle whose measure equals the measure of the part (I will be expressing all angle and arc measurements in radians). Each part and corresponding angle are:

$$m \ \mathrm{A} = m \ \mathrm{EDF} \qquad a = m \ \mathrm{FOE}$$

$$m \ \mathrm{B} = m \ \mathrm{DEF} \qquad b = m \ \mathrm{DOF}$$

$$m \ \mathrm{C} = m \ \mathrm{DFE} \qquad c = m \ \mathrm{DOE}$$

Because of this, $\sin a = \sin \ m \ \mathrm{FOE} = \dfrac{\mathrm{FE}}{\mathrm{OE}} = \dfrac{\mathrm{FE}}{\mathrm{ED}} \dfrac{\mathrm{ED}}{\mathrm{OE}} = \sin \mathrm{A} \sin c$.

Using similar logic and constructing a plane perpendicular to OB instead of OA, six more formulas are obtained for a total of seven. Using these seven formulas an additional three formulas can be derived to make a total of ten formulas. These ten formulas are:

$$\sin a = \sin \mathrm{A} \sin c \qquad \sin b = \sin c \sin \mathrm{B}$$

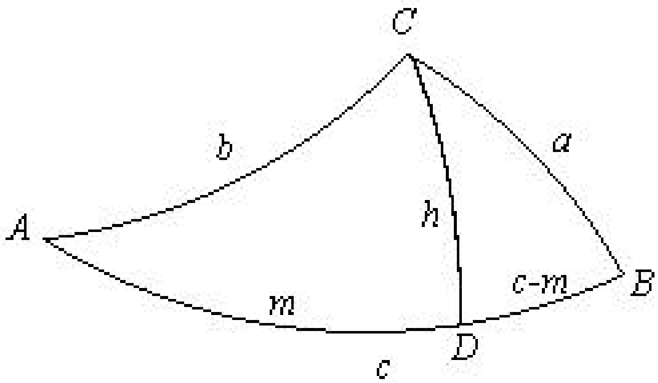
$$\tan a = \sin b \tan \mathrm{A} \qquad \tan b = \sin a \tan \mathrm{B}$$

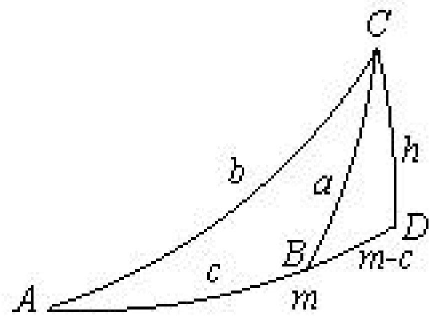
$$\tan a = \tan c \cos \mathrm{B} \qquad \tan b = \tan c \cos \mathrm{A}$$

$$\cos c = \cos a \cos b \qquad \cos c = \cot \mathrm{A} \cot \mathrm{B}$$

$$\cos \mathrm{A} = \cos a \cos \mathrm{B} \qquad \cos \mathrm{B} = \cos b \sin \mathrm{A}$$ [2]

To derive the law of sines and the law of cosines for sides, the construction of an altitude must be made. Construct the altitude from vertex C to side *c* and extend if necessary. The two cases (not extending side *c* and extending side *c*) a are depicted here:





Case I: *c* does not need to be extended. Case II: *c* does need to be extended.[3]

Law of Sines

The identities of right spherical triangles listed before yield:

$\sin h = \sin a \sin B$, and $\sin h = \sin b \sin A$

Substituting for $\sin h$, $\sin a \sin B = \sin b \sin A$. Dividing both sides by $\sin A \sin B$,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}.$$

If the altitude *h* were constructed from vertex B to side *b*, the derived identity would be:

$$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C}$$

By transitivity,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$ [4]

Law of Cosines for Sides

In both cases, ΔADC is right with D being the right angle. Applying identities for right spherical triangles,

$\cos a = \cos h \cos(c-m)$ for the first case, and

$\cos a = \cos h \cos(m-c)$ for the second case.

Because $\cos(c-m) = \cos(m-c)$, $\cos h \cos(c-m) = \cos h \cos(m-c)$.

Therefore, the first case can be used to derive the formula for both cases.

Because $\cos(\alpha-\beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$,

$\cos h \cos(c-m) = \cos h (\cos c \cos m + \sin c \sin m)$

Applying the identities for right spherical triangles to ΔADC,

$\cos b = \cos h \cos m$ (dividing both sides by $\cos h$ gives $\cos m = \dfrac{\cos b}{\cos h}$),

$\sin m = \tan h \cot A$, and $\sin h = \sin b \sin A$.

Substituting for $\cos m$ and $\sin m$,

$$\cos h (\cos c \cos m + \sin c \sin m) = \cos h \left(\cos c \, \frac{\cos b}{\cos h} + \sin c \tan h \cot A\right)$$

Distributing,

$$\cos h \left(\cos c \, \frac{\cos b}{\cos h} + \sin c \tan h \cot A\right) = \cos c \cos b + \sin c \sin h \cot A$$

Substituting $\sin h = \sin b \sin A$ and using the fact that $\sin A \cot A = \cos A$,

$\cos c \cos b + \sin c \sin h \cot A = \cos c \cos b + \sin c \sin b \cos A$.

Therefore, by transitivity,

$\cos a = \cos c \cos b + \sin c \sin b \cos A$.

By constructing the altitude h to sides a and b instead of c, these three forms of the law of cosines for sides can be derived:

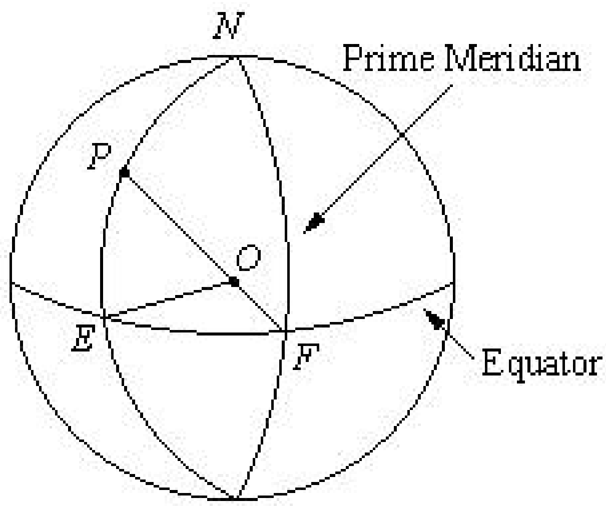
$$\cos a = \cos c \cos b + \sin c \sin b \cos A$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$

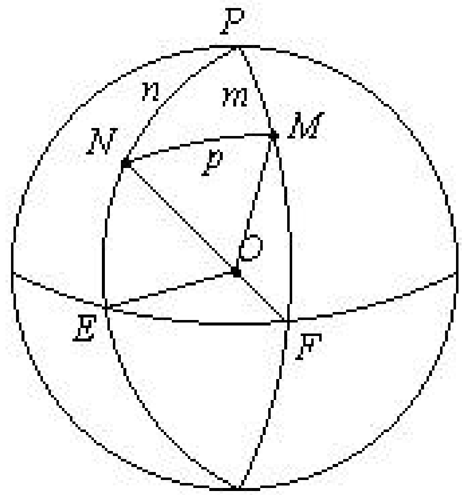
$$\cos c = \cos b \cos a + \sin b \sin a \cos C$$ [5]

For an example, I will use the triangle formed on the Earth by New York, Moscow and the North Pole. The distance between New York and Moscow will be unknown.

The Earth can be treated as a sphere with a radius of 3959mi. Such a sphere is called the terrestrial sphere. On the terrestrial sphere, the equator is the great circle that determines a plane perpendicular to the line drawn between the North Pole and the South Pole. A meridian is a great circle that intersects both poles. The prime meridian is the meridian that Greenwich, England intersects. In the following picture, the North Pole is at point N and the center of the sphere is at point O.



The longitude of a point P on the sphere is the measure of the arc intercepted on the equator by the meridian that P is on and the prime meridian. In the above picture, it is the measure of arc EF (which equals m EOF). Longitude is positive when P is West of the prime meridian and negative for when P is East of the prime meridian. The latitude of a point P on the sphere is the measure of the arc intercepted on the meridian that P is on by P and the equator. In the above picture, it is the measure of arc PE (which equals m POE). Latitude is positive when P is North of the equator and negative when P is South of the equator. The colattitude of a point P is $\pi/2$ - the latitude of point P. This is the measure of the arc intercepted on the meridian P is on by P and the North Pole (the measure of arc PN).



In the picture above, N is New York, M is Moscow, and P is the North Pole. The distance between the North Pole and New York is New York's colattitude (NP), and the distance between the North Pole and Moscow is Moscow's colattitude (MP). The difference in longitude between New York and Moscow is the arc intercepted on the equator by the meridians they are on. In the above picture, this is arc EF. The measure of arc EF = m EOF. Because the plane the equator is in is perpendicular to the line determined by the North Pole and South Pole, m EOF equals the measure of the dihedral angle formed by plane NEO and plane MFO (namely, N-PO-M). Because m EOF equals the measure of dihedral angle N-PO-M and the measure of N-PO-M is the measure of spherical angle NPM, m EOF equals the measure of spherical angle NPM. m EOF equals the measure of arc EF, which is the difference in longitude between New York and Moscow.

Therefore, the measure of the angle formed by New York and Moscow with North Pole (the North Pole as the vertex) is the difference in longitude between the two.

New York's longitude is $37\pi/90$, and its latitude is $2443\pi/10800$. Moscow's longitude is $-1127\pi/5400$ and latitude is $223\pi/720$.

n = colattitude of N = $\pi/2 - 2443\pi/10800 = 2957\pi/10800$.

m = colattitude of M = $\pi/2 - 223\pi/720 = 137\pi/720$.

P = difference in longitude = $37\pi/90 + 1127\pi/5400 = 3347\pi/5400$.

By the law of cosines for sides,

$\cos NM = \cos m \cos n + \sin m \sin n \cos P = .382398175$, so

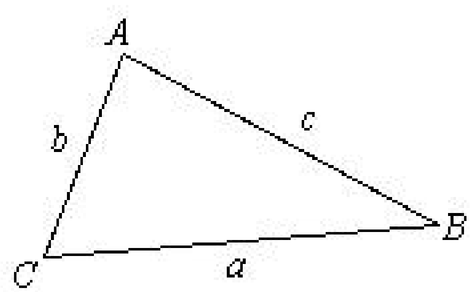
NM = Arccos .382398175 = 1.178405986 radians. To get the result in miles, this is multiplied by the radius (3959mi) to get 4665.309297mi. If one used Euclidean trigonometry to calculate this,

$\mathrm{NM}^2 = m^2 + n^2 - 2mn\cos \mathrm{P} = 23122014.49$, so

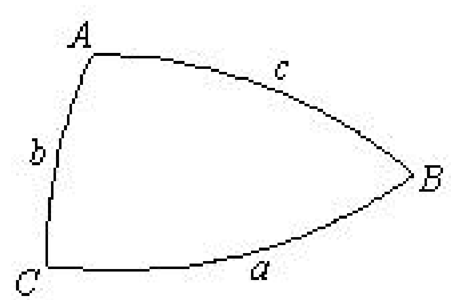
$\mathrm{NM} = \sqrt{23122014.49} = 4808.535587$ mi.

The result from using Euclidean trigonometry is significantly larger than the result from spherical trigonometry. This is because the given lengths wrap along the sphere, making the distance between them smaller than if they were straight.

The problem I presented at the beginning of the paper is: For a triangle ABC — where *a*, *b*, and *c* are the lengths of the sides opposite angles A, B, and C, respectively —, given *a*, *b*, and the measure of angle C, what is the relationship between *c* where ΔABC is in a Euclidean plane, and *c* where ΔABC is on the surface of a sphere and what is the relationship between the area of ΔABC in a Euclidean plane and ΔABC on the surface of a sphere? In other words, find $\frac{c_{\text{Euclidean}}}{c_{\text{spherical}}}$ and $\frac{\text{Area}_{\text{Euclidean}}}{\text{Area}_{\text{spherical}}}$. The triangles in this problem are illustrated here:



Euclidian



Spherical

The first part of the problem is to find $\frac{c_{\text{Euclidean}}}{c_{\text{spherical}}}$. The first part of finding this is to find the Euclidean *c* and spherical *c* independently.

Euclidean

$$c^2 = a^2 + b^2 - 2ab\cos \mathrm{C}$$
$$\Rightarrow c = \sqrt{a^2 + b^2 - 2ab\cos \mathrm{C}}$$

Spherical

I will use the constant *r* as the radius of the sphere. To apply law of cosines for sides, everything must be expressed in arc length.

$$\frac{\text{length}}{2\pi r} = \frac{\text{arc}}{2\pi}$$
$$\text{arc} = \frac{\text{length}}{r}$$

Applying the law of cosines for sides,

$$\cos\frac{c}{r} = \cos\frac{a}{r}\cos\frac{b}{r} + \sin\frac{a}{r}\sin\frac{b}{r}\cos C$$
$$\frac{c}{r} = \text{Arccos}\left[\cos\frac{a}{r}\cos\frac{b}{r} + \sin\frac{a}{r}\sin\frac{b}{r}\cos C\right]$$
$$c = r\,\text{Arccos}\left[\cos\frac{a}{r}\cos\frac{b}{r} + \sin\frac{a}{r}\sin\frac{b}{r}\cos C\right]$$

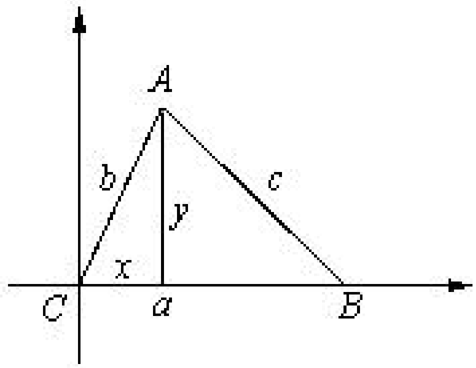
Therefore,

$$\frac{c_{\text{Euclidean}}}{c_{\text{spherical}}} = \frac{\sqrt{a^2 + b^2 - 2ab\cos C}}{r\,\text{Arccos}\left[\cos\frac{a}{r}\cos\frac{b}{r} + \sin\frac{a}{r}\sin\frac{b}{r}\cos C\right]}$$

The second part of the problem is to find $\frac{\text{Area}_{\text{Euclidean}}}{\text{Area}_{\text{spherical}}}$. The first part of finding this is to find each area independently.

<u>Euclidean</u>

Put ΔABC in a coordinate plane, with C at the origin.



sin C = $\frac{y}{b}$. à $b\sin C = y$. Area = ($\frac{1}{2}$)(base)(height), so Area = $\frac{1}{2}ay$.

Substituting for y,

$$\text{Area} = \frac{1}{2}ab\sin C$$

<u>Spherical</u>

Using to law of sines to get A and B, $\frac{\sin A}{\sin \frac{a}{r}} = \frac{\sin B}{\sin \frac{b}{r}} = \frac{\sin C}{\sin \frac{c}{r}}$. Therefore,

$$A = \text{Arcsin}\left(\frac{\sin \frac{a}{r} \sin C}{\sin \frac{c}{r}}\right) \text{ and } B = \text{Arcsin}\left(\frac{\sin \frac{b}{r} \sin C}{\sin \frac{c}{r}}\right)$$

The area of a spherical triangle is equal to r^2E, where E is the angle excess of the triangle.[6] The angle excess of a triangle is the amount the sum of the measures of its angles is over π (the sum of the angles - π). For example, the 90°-90°-90° triangle (whose angles are all 90°, or π/2 radians) has an area 1/8 of the surface area of the sphere. Using the angle excess formula, its area = $\frac{r^2\pi}{2}$. The surface area of a sphere is $4\pi r^2$, so the

result from the angle excess formula is true.

In the original problem, the angle excess formula gives:

Area = $r^2\left(A + B + C - \pi\right)$. Substituting for A and B,

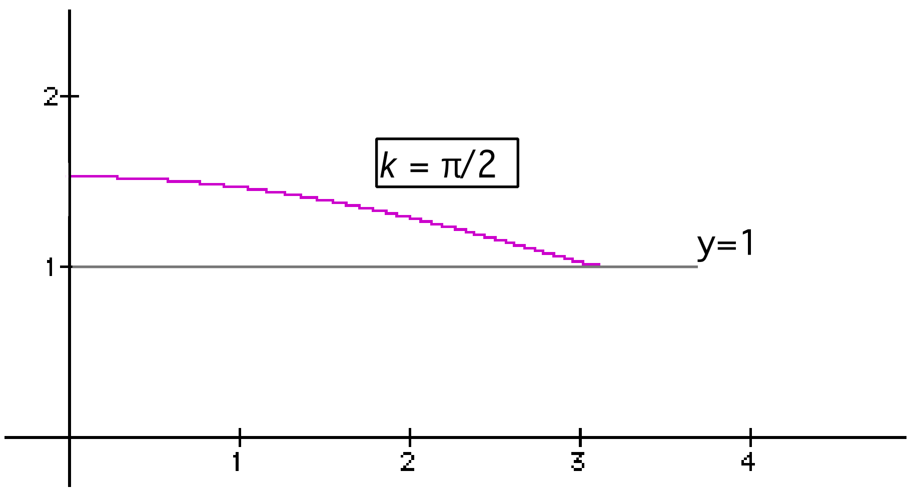
$$\text{Area} = r^2\left[\operatorname{Arcsin}\left(\frac{\sin\frac{a}{r}\ \sin C}{\sin\frac{c}{r}}\right) + \operatorname{Arcsin}\left(\frac{\sin\frac{b}{r}\ \sin C}{\sin\frac{c}{r}}\right) + C - \pi\right]$$

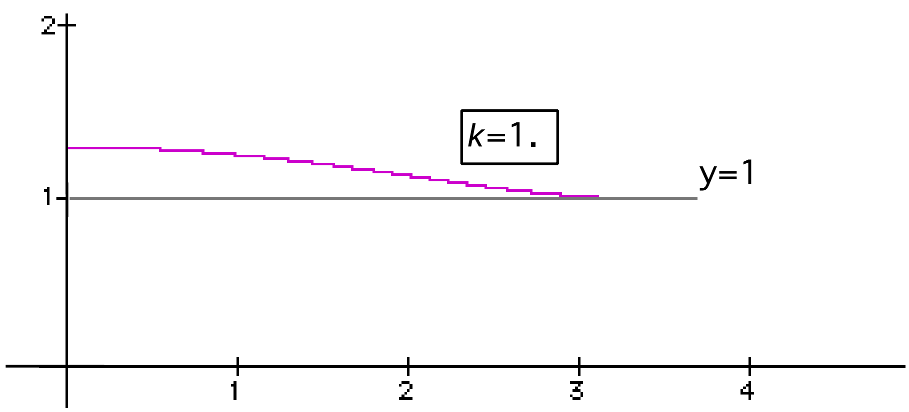
Therefore,

$$\frac{\text{Area}_{\text{Euclidean}}}{\text{Area}_{\text{spherical}}} = \frac{\frac{1}{2}ab\sin \mathrm{C}}{r^2\left[\operatorname{Arcsin}\left(\frac{\sin\frac{a}{r}\ \sin C}{\sin\frac{c}{r}}\right) + \operatorname{Arcsin}\left(\frac{\sin\frac{b}{r}\ \sin C}{\sin\frac{c}{r}}\right) + C - \pi\right]}$$

I will explore these ratios for the special case where $a=b=k$ on the unit sphere (where the radius, r, equals one). Because the only triangles I am dealing with are those whose angles each have a measure less than π and greater than 0, $0 < C < \pi$. Also, $0 < k < \pi$ because otherwise k would not fit the definition of a line segment. Below are graphs of the $\frac{c_{\text{Euclidean}}}{c_{\text{spherical}}}$ ratio as a function of C ($\frac{c_{\text{Euclidean}}}{c_{\text{spherical}}}$ is on the y-axis and C is on the x-axis).

There are two graphs: one where $k=\pi/2$ and another where $k=1.2$.





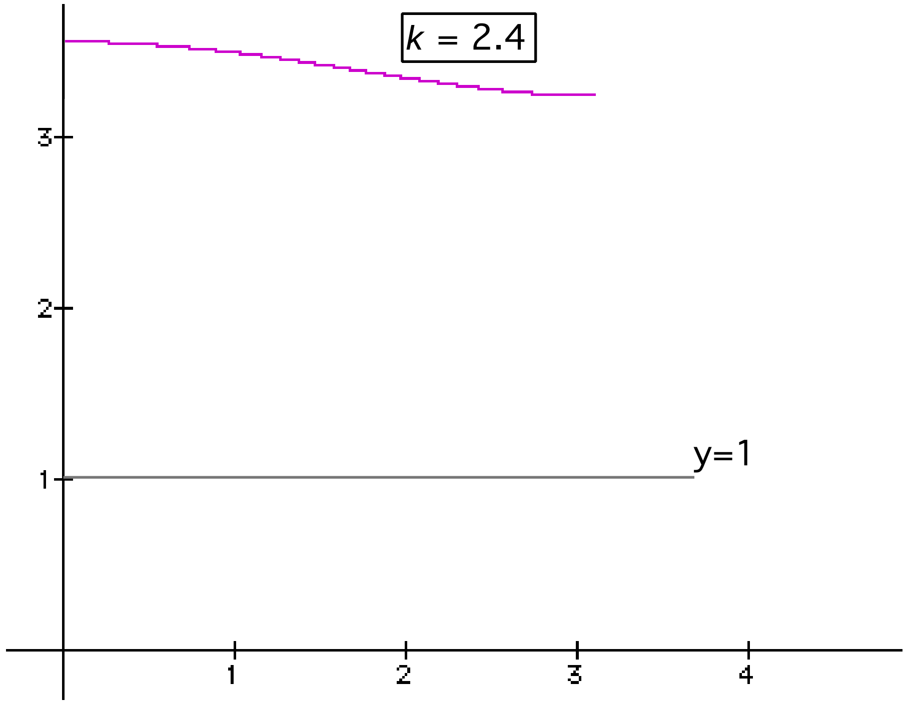
As k increases, $\frac{c_{\text{Euclidean}}}{c_{\text{spherical}}}$ increases (no matter what the value of C is).

This implies that the spherical *c* becomes increasingly smaller than the Euclidean *c* as more of the sphere is covered by sides *a* and *b*.

The limit of $\frac{c_{\text{Euclidean}}}{c_{\text{spherical}}}$ as *k* approaches zero is 1. This implies that for small values of *k*, the difference between Euclidean *c* and spherical *c* is negligible. An example of this is when one deals with local triangles on the Earth. If one is dealing with triangles that are small relative to the size of the Earth (these are about all of the triangles encountered in every-day life), one can use Euclidean trigonometry and get results just about as accurate as one would have gotten had they used spherical trigonometry.

The limit of $\frac{c_{\text{Euclidean}}}{c_{\text{spherical}}}$ as C approaches π is 1. This is because the length of side *c* is getting closer to $a+b$ (which always has the same value in the Euclidean triangle as in the spherical triangle).

When $k > \pi/2$, there is a "liftoff" of the entire graph. The graph of $\frac{c_{\text{Euclidean}}}{c_{\text{spherical}}}$ where $k=2.4$ (a number greater than $\pi/2$) is presented below:



This liftoff is due to a wrap-around effect. Wrap-around occurs when k is greater than one quarter of the circumference of the sphere. When this happens, the length of side c decreases as k increases because sides a and and b are wrapping around the place where c is maximized. Thus, the maximum value for c is when $k=\pi/2$. My proof of this is presented here:

$$c = \operatorname{Arccos}\left[\cos^2 k + \sin^2 k \cos C\right]$$

Differentiating with respect to k yields:

$$\frac{dc}{dk} = \left(\frac{-1}{\sqrt{1-\left[\cos^2 k + \sin^2 k \cos C\right]^2}}\right)\left[-2\cos k \sin k + 2\sin k \cos k \cos C\right]$$

Factoring out $2\cos k \sin k$,

$$\frac{\mathrm{d}c}{\mathrm{d}k} = \frac{-1}{\sqrt{1-\left[\cos^2 k + \sin^2 k \cos C\right]^2}}\left[2\cos k \sin k \left[\cos C - 1\right]\right]$$

$$\frac{\mathrm{d}c}{\mathrm{d}k} = \frac{-2\cos k \sin k \left[\cos C - 1\right]}{\sqrt{1-\left[\cos^2 k + \sin^2 k \cos C\right]^2}}$$

The critical points are at the values of *k* that make the derivative either equal to zero or undefined. The derivative equals zero when its numerator equals zero, and it is undefined when the denominator equals zero. Thus, the values of *k* that are critical points are values that make the numerator equal to zero or make the denominator equal to zero.

<u>Case I: the numerator equals zero</u>

Setting the numerator equal to zero,

$$-2\cos k \sin k \left[\cos C - 1\right] = 0$$

Both cos C - 1 and -2 are constants, so both sides of the equation can be divided by $-2\left[\cos C - 1\right]$ without losing a root. Executing this division,

$$\cos k \sin k = 0$$

If a product of two numbers equals zero, than one or both must equal zero. Therefore,

$$\cos k = 0 \quad \text{or} \quad \sin k = 0$$
$$\Rightarrow k = \frac{\pi}{2} \quad \text{or} \quad k = 0 \quad \text{or} \quad k = \pi$$

Case II: the denominator equals zero

Setting the denominator equal to zero,

$$\sqrt{1 - \left[\cos^2 k + \sin^2 k \cos C\right]^2} = 0$$

Squaring both sides, adding $\left[\cos^2 k + \sin^2 k \cos C\right]^2$ to both sides, and then taking the square root of both sides,

$$\cos^2 k + \sin^2 k \cos C = \pm 1$$

Substituting $1 - \cos^2 k$ for $\sin^2 k$,

$$\cos^2 k + \left[1 - \cos^2 k\right]\cos C = \pm 1$$

Distributing cos C,

$$\cos^2 k + \cos C - \cos^2 k \cos C = \pm 1$$

Factoring out $\cos^2 k$,

$$\left[1 - \cos C\right]\cos^2 k + \cos C = \pm 1$$

Subtracting cos C from both sides and dividing both sides by (1 - cos C),

$$\cos^2 k = \frac{\pm 1 - \cos C}{1 - \cos C}$$

$$\Rightarrow k = \operatorname{Arccos}\left(\pm\sqrt{\frac{\pm 1 - \cos C}{1 - \cos C}}\right)$$

Because the radicand must be positive and |cos C| < 1, the -1 of the ± 1 must be discarded. Therefore,

$$k = \operatorname{Arccos}\left(\pm\sqrt{\frac{1 - \cos C}{1 - \cos C}}\right)$$

$$\Rightarrow k = \operatorname{Arccos}\left(\pm\sqrt{1}\right) = \operatorname{Arccos}\left(\pm 1\right)$$

$$\Rightarrow k = 0 \quad \text{or} \quad k = \pi$$

The set of all the critical points is the union of the points from Case I (numerator = 0) and the points from Case II (denominator = 0).

Therefore, the critical points are:

$$k = \frac{\pi}{2},\ k = 0, \text{ and } k = \pi$$

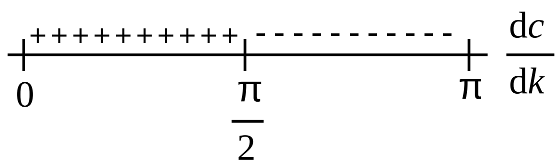
To find which are relative maxima, I will make a sign diagram of $\frac{dc}{dk}$. $\frac{dc}{dk}$ is reprinted here:

$$\frac{dc}{dk} = \frac{-2\cos k \sin k\,(\cos C - 1)}{\sqrt{1 - (\cos^2 k + \sin^2 k \cos C)^2}}$$

(cos C - 1) is always negative because |cos C| < 1. The denominator is always greater than or equal to zero because it is a positive square root.-2 is always negative, so the sign of $\frac{dc}{dk}$ when k is not a critical point is the sign of $\cos k \sin k$.

When $0 < k < \frac{\pi}{2}$, $\cos k$ and $\sin k$ are both positive, so $\frac{dc}{dk}$ is positive.

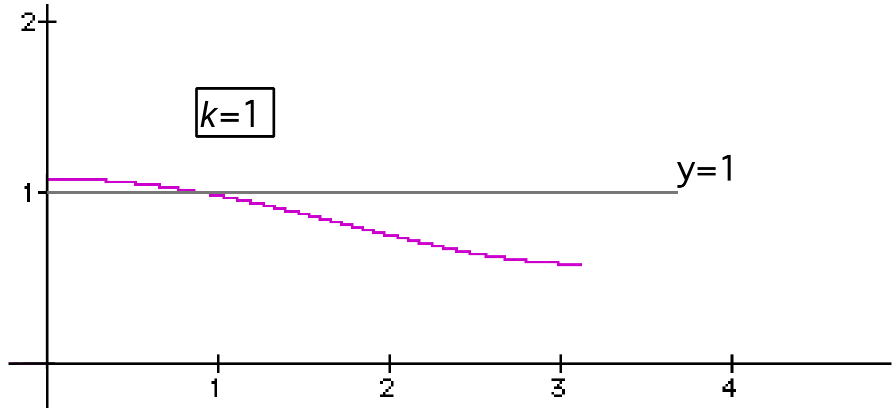
When $\frac{\pi}{2} < k < \pi$, $\cos k$ is negative and $\sin k$ is positive, so $\frac{\mathrm{d}c}{\mathrm{d}k}$ is negative when $\frac{\pi}{2} < k < \pi$. The sign diagram is drawn below:

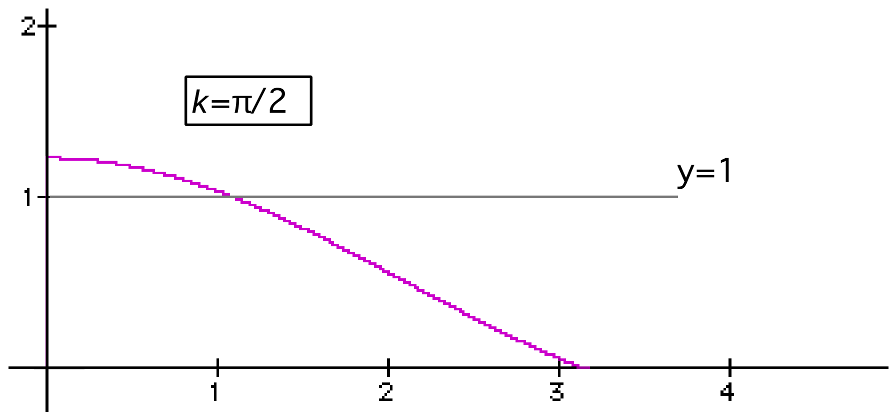


Thus, c is increasing over the interval (0, π/2) and decreasing over the interval (π/2, π). Therefore, there is a relative maximum at $k = \frac{\pi}{2}$.

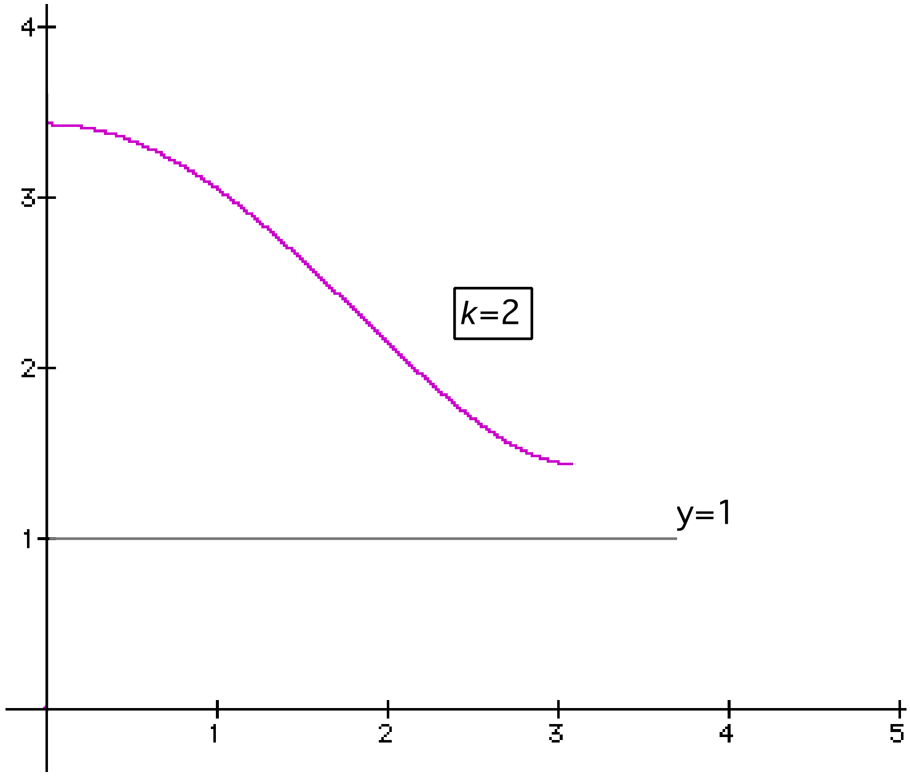
Within the domain there is no other relative maximum, so $k = \frac{\pi}{2}$ is the absolute maximum for c.

Below are two graphs of $\frac{\mathrm{Area}_{\mathrm{Euclidean}}}{\mathrm{Area}_{\mathrm{spherical}}}$ as a function of C. $\frac{\mathrm{Area}_{\mathrm{Euclidean}}}{\mathrm{Area}_{\mathrm{spherical}}}$ is on the y-axis, and C is on the x-axis. In one graph, $k=1$. In the other, $k=\pi/2$.





The limit as k approaches zero of $\frac{\text{Area}_{\text{Euclidean}}}{\text{Area}_{\text{spherical}}}$ equals one. As k increases, the y-intercept increases. Also as k increases, the limit as C approaches π of $\frac{\text{Area}_{\text{Euclidean}}}{\text{Area}_{\text{spherical}}}$ decreases, and reaches a minimum of zero at $k = \pi/2$. When $k > \pi/2$, the wrap-around effect creates a liftoff of the graph. A graph where $k > \pi/2$ is presented here:



In the amount of time I was given to complete this paper, I was unable to accomplish several things. I could not explore the $\frac{\text{Area}_{\text{Euclidean}}}{\text{Area}_{\text{spherical}}}$ ratio for isosceles triangles as much as I wanted. Also, I did not have enough time to explore the two ratios for non-isosceles triangles. I believe wraparound affects the ratios for non-isosceles triangles when the sum $a+b$ is greater than $\pi r/2$ ($\pi/2$ on the unit sphere). In addition, I was unable to explore the ratios when the radius of the sphere is a constant r. However, the work I did where the radius equals one can apply to all spheres; the units of length can always be defined such that the length of the radius of the sphere equals one. For example, on Earth one can measure everything in terms of Earth radii.

NOTES

1. Picture from Paul R. Rider, *Plane and Spherical Trigonometry*, New York: The Macmillan Company, p. 201

2. Ten formulas from Rider, *Plane and Spherical Trigonometry* p. 203

3. Both pictures from Rider, *Plane and Spherical Trigonometry* p. 212

4. Derivation adapted from Rider, *Plane and Spherical Trigonometry* pp. 211-212.

5. Derivation adapted from Rider, *Plane and Spherical Trigonometry* pp. 212-213

6. Angle excess formula adapted from Rider, *Plane and Spherical Trigonometry* p. 200